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AG II - Schemes and sheaves

Exercises – week 5

Exercise 1. *Closed subschemes of* Proj. Let B be an $\mathbb{N}$-graded ring and I be a homogeneous ideal. Show that the subset

$$V_+(I) = \{\mathfrak{p} \in \mathrm{Proj}(B) \mid I \subset \mathfrak{p}\}$$

is closed in $\mathrm{Proj}(B)$ and can be endowed a scheme structure *via* $\mathrm{Proj}(B/I)$. Show that for any $b \in B_+$ homogeneous

$$(B/I)_{(b)} \cong B_{(b)}/I_{(b)}.$$

In what follows, $V_+(I)$ is taken in the schematic sense of the previous exercise.

Exercise 2. Proj *and base change.* Let R be a ring and R' be an R-algebra. Let A be an $\mathbb{N}$-graded ring such that A_0 is an R-algebra. Define $B = A \otimes_R R'$ and write $\psi \colon A \to B$.

(1) With the notations of *week 4, exercise 5* show that $U(\psi) = \mathrm{Proj}(B)$.
(2) Show that the commutative diagram

$$\begin{tikzcd} \mathrm{Proj}(B) \arrow[r] \arrow[d] & \mathrm{Proj}(A) \arrow[d] \\ \mathrm{Spec}(R') \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

is a Cartesian square.[1]
(3) Let I be an ideal of R. Show that if $R' = R/I$ then we have $\mathrm{Proj}(B) = V_+(\bigoplus_{n \geq 0} IA_n)$ inside $\mathrm{Proj}(A)$.
(4) Let R be a ring. Let $R' = R[x_0, \dots, x_n]$. Using the above, realize $\mathbb{P}^n_R \times_R \mathbb{A}^{n+1}_R$ as the Proj of a graded ring.

Exercise 3. *Blow-ups.* Let R be a ring and $I \subset R$ an ideal. We define the blow-up of $\mathrm{Spec}(R)$ at $V(I)$ to be the map ($I^0 = R$)

$$b \colon \mathrm{Bl}_I = \mathrm{Proj}(\bigoplus_{n \geq 0} I^n) \to \mathrm{Spec}(R).$$

[1] It means that for every scheme X and pair of morphisms $f_1 \colon X \to \mathrm{Proj}(A)$ and $f_2 \colon X \to \mathrm{Spec}(R')$ that agree when further composing to $\mathrm{Spec}(R)$, then there exists a unique morphism to $f \colon X \to \mathrm{Proj}(B)$ such that f is f_1 and f_2 when postcomposing with the maps to $\mathrm{Proj}(A)$ and $\mathrm{Spec}(R')$ respectively.

The *exceptional divisor* of the blow-up is the closed subscheme of $\mathrm{Proj}(\bigoplus_{n\geq 0} I^n)$

$$E = V_+(\bigoplus_{n\geq 0} I^{n+1}).$$

(1) Show that b defines an isomorphism of schemes

$$b\colon \mathrm{Bl}_I \setminus E \to \mathrm{Spec}(R) \setminus V(I).$$

(2) Let A be a ring, $R = A[x_0, \ldots, x_n]$ and $I = (x_0, \ldots, x_n)$. Show that $E \cong \mathbb{P}^n_A$.

Remark. Let us introduce a bit of intuition. Points (1) and (2) subsume the key philosophy of blow-ups. First of all a blow-up is a map which is an isomorphism outside of a the fiber closed subscheme $V(I)$. We will see later that in nice cases I/I^2 is to be interpreted as the *conormal bundle* of $V(I)$ in $\mathrm{Spec}(R)$ (*i.e.* tangent vectors going out of $V(I)$). Therefore E can be interpreted as the projective space of the vector space of directions outside of $V(I)$. Meaning that for each direction going outside of $V(I)$, there is a corresponding point in E. For example, in the actual computation above, the exceptional divisor E is the space of lines through the origin in $\mathbb{A}^{n+1}$, i.e. $\mathbb{P}^n$.

(3) *Standard blow-up charts.* Consider the same setting as in the last item. Show that Bl_I can be identified as the scheme

$$V_+(x_iY_j - x_jY_i)$$

inside $\mathbb{A}^{n+1}_A \times \mathbb{P}^n_A = \mathrm{Proj}(A[x_0, \ldots, x_n] \otimes_A A[Y_0, \ldots, Y_n])$ where the grading is taken to be the Y-grading (see exercise 2.4).

Remark. See here for a representation of the blow-up at (x_0, x_1) of $\mathbb{A}^2$ using the standard charts. The projection to the x, y-plane is a bijection outside of the pre-image of the origin which is a line.

Exercise 4. *Strict transforms.* Let R be a ring. Let I be an ideal and $b\colon \mathrm{Bl}_I \to \mathrm{Spec}(R)$ the blow-up at the ideal I. Let $J \subset R$ be another ideal. We define the *strict transform of* $V(J)$ to be the blow-up of $V((I + J)/J)$ in $\mathrm{Spec}(R/J)$.

(1) Show that St_J can be identified with be the closed subscheme of Bl_I

$$V_+\left(\bigoplus_n I^n \cap J\right).$$

(2) Show that b induces an isomorphism

$$b\colon \mathrm{St}_J \setminus E \to V(J) \setminus V(I).$$

(3) *Resolving a singularity.* Let k be a field. Compute the strict transform with $R = k[x_0, x_1]$, the ideal $I = (x_0, x_1)$ and $J = (x_1^2 - (x_0^3 + x_0^2))$. Use the standard blow-up charts. Show that this strict transform is regular.

Remarks. The equation of the last item is the equation of a *nodal curve* which is a type of singular curve. See here for a representation. The tangent space at the origin has dimension 2, which is why the curve is not regular.

Since the blow-up at a point replaces a point by "directions out of the point", it is no surprise that blowing up a node at its nodal point removes the singularity.

Exercise 5. *A criterion for affineness.* Let X be a scheme. Let $f \in \mathcal{O}_X(X)$. Let

$$X_f = \{x \in X \mid f(x) \neq 0\}$$

where $f(x)$ denotes the image of f in $k(x)$.

(1) Show that X_f is open.
(2) Show the following lemma.
Lemma. *A scheme X is affine if and only if there exists $f_1, \ldots, f_n \in \mathcal{O}_X(X) = A$ that generates the unit ideal in A and opens X_{f_i} are affine.*

For item (2), show that it suffices to show that the natural map

$$\Gamma(X, \mathcal{O}_X)_{f_i} \to \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}})$$

is an isomorphism.
To show this, show the following intermediate lemmas.

(a) For injectivity, show that if X is a quasi compact scheme and $g \in \Gamma(X, \mathcal{O}_X)$ such that the restriction of g to $\Gamma(X_f, \mathcal{O}_{X_f})$ is zero, then there is some $n > 0$ such that $f^n g = 0$ in $\Gamma(X, \mathcal{O}_X)$.
(b) For surjectivity, show that if X admits a finite cover by affine schemes with intersections being quasi-compact, then if $g \in \Gamma(X_f, \mathcal{O}_{X_f})$ then there is some $n > 0$ such that $f^n g$ is the image of an element $\Gamma(X, \mathcal{O}_X)$.

Exercise 6. Let k be an algebraically closed field. Are the following schemes regular?

(1) $V_+(XZ - Y^2) \subseteq \mathbb{P}^2$
(2) $V(xz - y^2) \subseteq \mathbb{A}^3$
(3) $V_+(XZ - Y^2, YW - Z^2, XW - YZ) \subseteq \mathbb{P}^3$
(4) $V(y^2 - x(x-1)(x+1) \subseteq \mathbb{A}^2$

Hint: Be careful about the characteristic of k!

Exercise 7. Let $k = \mathbb{F}_p(t)$, and consider $X_k = V(x^p - t) \subseteq \mathbb{A}^1_k$. Show that X is regular.
However, let $k' = \mathbb{F}_p(t^{1/p}) \supseteq k$. Then show that $X_{k'}$ is not regular.

Remark. As we will shortly see, we have a Cartesian diagram

$$\begin{tikzcd} X_{k'} \arrow[r] \arrow[d] & X_k \arrow[d] \\ \mathrm{Spec}(k') \arrow[r] & \mathrm{Spec}(k). \end{tikzcd}$$

We say that $X_{k'}$ is obtained by a *base field extension* (or *base change*) of X_k.
What the previous exercise then tells us is that that being regular (or even reduced) is not stable under field extensions. We will later see a notion, called *smoothness*, which implies regularity, and which is stable under base change.